Single Subject Matter Standards:
Mathematics

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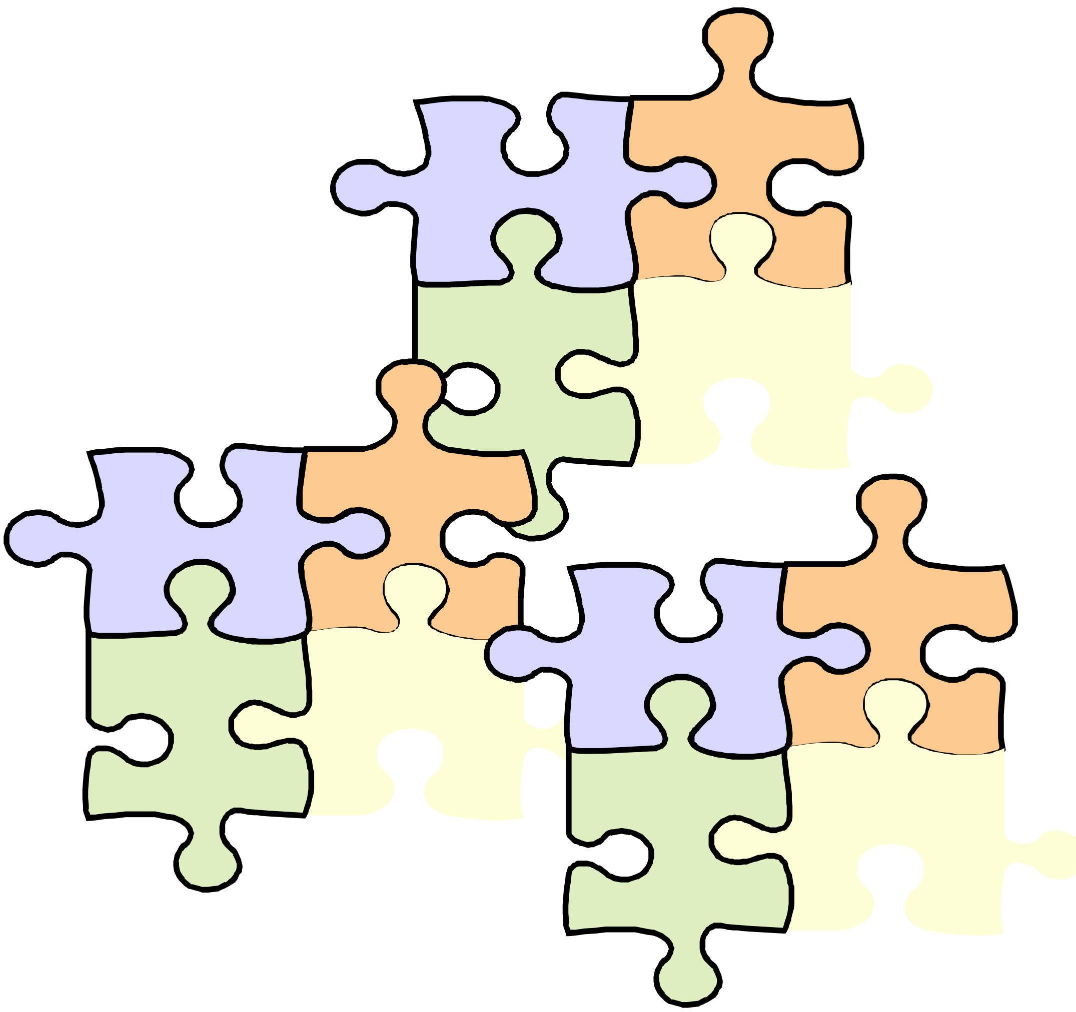
California Commission on Teacher Credentialing



Single Subject Matter Standards of Quality and Effectiveness for Programs in Mathematics

California Commission on Teacher Credentialing

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Standards Common to All

Standard 1: Program Philosophy and Purpose

The subject matter preparation program is based on an explicit statement of program philosophy that expresses its purpose, design, and desired outcomes in relation to the Standards of Quality and Effectiveness for Single Subject Teaching Credential Programs. The program provides the coursework and field experiences necessary to teach the specified subject to all of California's diverse public school population. Subject matter preparation in the program for prospective teachers is academically rigorous and intellectually stimulating. The program curriculum reflects and builds on the State-adopted *Academic Content Standards for K-12 Students* and *Curriculum Frameworks for California Public Schools.* The program is designed to establish a strong foundation in and understanding of subject matter knowledge for prospective teachers that provides a basis for continued development during each teacher's professional career. The sponsoring institution assigns high priority to and appropriately supports the program as an essential part of its mission.

<u>Required Elements</u>

1.1 The program philosophy, design, and intended outcomes are consistent with the content of the State-adopted Academic Content Standards for K-12 students and Curriculum Frameworks for California public schools.

1.2 The statement of program philosophy shows a clear understanding of the preparation that prospective teachers need in order to be effective in delivering academic content to all students in California schools.

1.3 The program provides prospective teachers with the opportunity to learn and apply significant ideas, structures, methods and core concepts in the specified subject discipline(s) that underlies the 6-12 curriculum.

1.4 The program prepares prospective single-subject teachers to analyze complex discipline-based issues; synthesize information from multiple sources and perspectives; communicate skillfully in oral and written forms; and use appropriate technologies.

1.5 Program outcomes are defined clearly and assessments of prospective teachers and program reviews are appropriately aligned.

1.6 The institution conducts periodic review of the program philosophy, goals, design, and outcomes consistent with the following: campus program assessment timelines, procedures, and policies; ongoing research and thinking in the discipline; nationally accepted content standards and recommendations; and the changing needs of public schools in California.

Standard 2: Diversity and Equity

The subject matter program provides equitable opportunities to learn for all prospective teachers by utilizing instructional, advisement and curricular practices that insure equal access to program academic content and knowledge of career options. Included in the program are the essential understandings, knowledge and appreciation of the perspectives and contributions by and about diverse groups in the discipline.

<u>Required Elements:</u>

2.1 In accordance with the Education Code Chapter 587, Statutes of 1999, (See Appendix A), human differences and similarities to be examined in the program include, but are not limited to those of sex, race, ethnicity, socio-economic status, religion, sexual orientation, and exceptionality. The program may also include study of other human similarities and differences.

2.2 The institution recruits and provides information and advice to men and women prospective teachers from diverse backgrounds on requirements for admission to and completion of subject matter programs.

2.3 The curriculum in the Subject Matter Program reflects the perspectives and contributions of diverse groups from a variety of cultures to the disciplines of study.

2.4 In the subject matter program, classroom practices and instructional materials are designed to provide equitable access to the academic content of the program to prospective teachers from all backgrounds.

2.5 The subject matter program incorporates a wide variety of pedagogical and instructional approaches to academic learning suitable to a diverse population of prospective teachers. Instructional practices and materials used in the program support equitable access for all prospective teachers and take into account current knowledge of cognition and human learning theory.

Standard 3: Technology

The study and application of current and emerging technologies, with a focus on those used in K-12 schools, for gathering, analyzing, managing, processing, and presenting information is an integral component of each prospective teacher's program study. Prospective teachers are introduced to legal, ethical, and social issues related to technology. The program prepares prospective teachers to meet the current technology requirements for admission to an approved California professional teacher preparation program.

Required Elements**:**

3.1 The institution provides prospective teachers in the subject matter program access to a wide array of current technology resources. The program faculty selects these technologies on the basis of their effective and appropriate uses in the disciplines of the subject matter program.

3.2 Prospective teachers demonstrate information processing competency, including but not limited to the use of appropriate technologies and tools for research, problem solving, data acquisition and analysis, communications, and presentation.

3.3 In the program, prospective teachers use current and emerging technologies relevant to the disciplines of study to enhance their subject matter knowledge and understanding.

Standard 4: Literacy

The program of subject matter preparation for prospective Single Subject teachers develops skills in literacy and academic discourse in the academic disciplines of study. Coursework and field experiences in the program include reflective and analytic instructional activities that specifically address the use of language, content and discourse to extend meaning and knowledge about ideas and experiences in the fields or discipline of the subject matter.

Required Elements**:**

4.1 The program develops prospective teachers' abilities to use academic language, content, and disciplinary thinking in purposeful ways to analyze, synthesize and evaluate experiences and enhance understanding in the discipline.

4.2 The program prepares prospective teachers to understand and use appropriately academic and technical terminology and the research conventions of the disciplines of the subject matter.

4.3 The program provides prospective teachers with opportunities to learn and demonstrate competence in reading, writing, listening, speaking, communicating and reasoning in their fields or discipline of the subject matter.

Standard 5: Varied Teaching Strategies

In the program, prospective Single Subject teachers participate in a variety of learning experiences that model effective curriculum practices, instructional strategies and assessments that prospective teachers will be expected to use in their own classrooms.

Required Elements**:**

5.1 Program faculty include in their instruction a variety of curriculum design, classroom organizational strategies, activities, materials and field experiences incorporating observing, recording, analyzing and interpreting content as appropriate to the discipline.

5.2 Program faculty employ a variety of interactive, engaging teaching styles that develop and reinforce skills and concepts through open-ended activities such as direct instruction, discourse, demonstrations, individual and cooperative learning explorations, peer instruction and student-centered discussion.

5.3 Faculty development programs provide tangible support for subject matter faculty to explore and use exemplary and innovative curriculum practices.

5.4 Program faculty use varied and innovative teaching strategies, which provide opportunities for prospective teachers to learn how content is conceived and organized for instruction in a way that fosters conceptual understanding as well as procedural knowledge.

5.5 Program coursework and fieldwork include the examination and use of various kinds of technology that are appropriate to the subject matter discipline.

Standard 6: Early Field Experiences

The program provides prospective Single Subject teachers with planned, structured field experiences in departmentalized classrooms beginning as early as possible in the subject matter program. These classroom experiences are linked to program coursework and give a breadth of experiences across grade levels and with diverse populations. The early field experience program is planned collaboratively by subject matter faculty, teacher education faculty and representatives from school districts. The institution cooperates with school districts in selecting schools and classrooms for introductory classroom experiences. The program includes a clear process for documenting each prospective teacher's observations and experiences.

Required Elements**:**

6.1 Introductory experiences shall include one or more of the following activities: planned observations, instruction or tutoring experiences, and other school based observations or activities that are appropriate for undergraduate students in a subject matter preparation program.

6.2 Prospective teachers' early field experiences are substantively linked to the content of coursework in the program.

6.3 Fieldwork experiences for all prospective teachers include significant interactions with K-12 students from diverse populations represented in California public schools and cooperation with at least one carefully selected teacher certificated in the discipline of study.

6.4 Prospective teachers will have opportunities to reflect on and analyze their early field experiences in relation to course content. These opportunities may include field experience journals, portfolios, and discussions in the subject matter courses, among others.

6.5 Each prospective teacher is primarily responsible for documenting early field experiences. Documentation is reviewed as part of the program requirements.

Standard 7: Assessment of Subject Matter Competence

The program uses formative and summative multiple measures to assess the subject matter competence of each candidate. The scope and content of each candidate's assessment is consistent with the content of the subject matter requirements of the program and with institutional standards for program completion.

Required Elements**:**

7.1 Assessment within the program includes multiple measures such as student performances, presentations, research projects, portfolios, field experience journals, observations, and interviews as well as oral and written examinations based on criteria established by the institution.

7.2 The scope and content of each assessment is congruent with the specifications for the subject matter knowledge and competence as indicated in the content domains of the Commission-adopted subject matter requirement.

7.3 End-of-program summative assessment of subject matter competence includes a defined process that incorporates multiple measures for evaluation of performance.

7.4 Assessment scope, process, and criteria are clearly delineated and made available to students when they begin the program.

7.5 Program faculty regularly evaluate the quality, fairness, and effectiveness of the assessment process, including its consistency with program requirements.

7.6 The institution that sponsors the program determines, establishes and implements a standard of minimum scholarship (such as overall GPA, minimum course grade or other assessments) of program completion for prospective single subject teachers.

Standard 8: Advisement and Support

The subject matter program includes a system for identifying, advising and retaining prospective Single Subject teachers. This system will comprehensively address the distinct needs and interests of a range of prospective teachers, including resident prospective students, early deciders entering blended programs, groups underrepresented among current teachers, prospective teachers who transfer to the institution, and prospective teachers in career transition.

Required Elements:

8.1 The institution will develop and implement processes for identifying prospective Single Subject teachers and advising them about all program requirements and career options.

8.2 Advisement services will provide prospective teachers with information about their academic progress, including transfer agreements and alternative paths to a teaching credential, and describe the specific qualifications needed for each type of credential, including the teaching assignments it authorizes.

8.3 The subject matter program facilitates the transfer of prospective teachers between post-secondary institutions, including community colleges, through effective outreach and advising and the articulation of courses and requirements. The program sponsor works cooperatively with community colleges to ensure that subject matter coursework at feeder campuses is aligned with the relevant portions of the *State-adopted Academic Content Standards for K-12 Students in California Public Schools*.

8.4 The institution establishes clear and reasonable criteria and allocates sufficient time and personnel resources to enable qualified personnel to evaluate prospective teachers' previous coursework and/or fieldwork for meeting subject matter requirements.

Standard 9: Program Review and Evaluation

The institution implements a comprehensive, ongoing system for periodic review of and improvement to the subject matter program. The ongoing system of review and improvement involves university faculty, community college faculty, student candidates and appropriate public schools personnel involved in beginning teacher preparation and induction. Periodic reviews shall be conducted at intervals not exceeding 5 years.

<u>Required Elements:</u>

9.1 Each periodic review includes an examination of program goals, design, curriculum, requirements, student success, technology uses, advising services, assessment procedures and program outcomes for prospective teachers.

9.2 Each program review examines the quality and effectiveness of collaborative partnerships with secondary schools and community colleges.

9.3 The program uses appropriate methods to collect data to assess the subject matter program's strengths, weaknesses and areas that need improvement. Participants in the review include faculty members, current students, recent graduates, education faculty, employers, and appropriate community college and public school personnel.

9.4 Program improvements are based on the results of periodic reviews, the inclusion and implications of new knowledge about the subject(s) of study, the identified needs of program students and school districts in the region, and curriculum policies of the State of California.

Standard 10: Coordination

One or more faculty responsible for program planning, implementation and review coordinate the Single Subject Matter Preparation Program. The program sponsor allocates resources to support effective coordination and implementation of all aspects of the program. The coordinator(s) fosters and facilitates ongoing collaboration among academic program faculty, local school personnel, local community colleges and the professional education faculty.

<u>Required Elements:</u>

10.1 A program coordinator will be designated from among the academic program faculty.

10.2 The program coordinator provides opportunities for collaboration by faculty, students, and appropriate public school personnel in the design and development of and revisions to the program, and communicates program goals to the campus community, other academic partners, school districts and the public.

10.3 The institution allocates sufficient time and resources for faculty coordination and staff support for development, implementation and revision of all aspects of the program.

10.4 The program provides opportunities for collaboration on curriculum development among program faculty.

10.5 University and program faculty cooperate with community colleges to coordinate courses and articulate course requirements for prospective teachers to facilitate transfer to a baccalaureate degree-granting institution.

Standards for Mathematics

Standard 11: Required Subjects of Study

In the program, each prospective teacher studies and learns advanced mathematics that incorporates the Mathematics Content Standards for California Public Schools: Kindergarten Through Grade Twelve (1997) and the Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve (1999). The curriculum of the program addresses the *Subject Matter Requirements* and standards of program quality as set forth in this document.[1]

Required Elements:

11.1 Required coursework includes the following major subject areas of study: algebra, geometry, number theory, calculus, history of mathematics, and statistics and probability. This coursework also incorporates the content of the student academic content standards from an advanced viewpoint (see *Attachment to Standard 11: Required Subjects of Study* page 18). Furthermore, infused in required coursework are connections to the middle school and high school curriculum.

11.2 Required coursework exposes underlying mathematical reasoning, explores connections among the branches of mathematics, and provides opportunities for problem solving and mathematical communication.

11.3 Required courses are applicable to the requirements for a major in mathematics. Remedial classes and other studies normally completed in K-12 schools are not counted in satisfaction of the required subjects of study.

11.4 The institution that sponsors the program determines, establishes and implements a standard of minimum scholarship for coursework in the program.

11.5 Required coursework includes work in computer science and/or related mathematics such as: 1) discrete structures (sets, logic, relations and functions) and their application in the design of data structures and programming; 2) design and analysis of algorithms including the use of recursion and combinations, and 3) use of the computer applications and other technologies to solve problems.

[1] The Subject Matter Requirements are complemented by the **Attachment to Standard 11**, starting on page 19

Standard 12: Problem Solving

In the program, prospective teachers of mathematics develop effective strategies for solving problems both within the discipline of mathematics and in applied settings that include non-routine situations. Problem-solving challenges occur throughout the program of subject matter preparation in mathematics. Through coursework in the program, prospective teachers develop a sense of inquiry and perseverance in solving problems.

<u>Required Elements:</u>

In the program, each prospective teacher learns and demonstrates the ability to:

12.1 Place mathematical problems in context and explore their relationship with other problems.

12.2 Solve mathematical problems in more than one way when possible.

12.3 Generalize mathematical problems in more than one way when possible.

12.4 Use appropriate technologies to conduct investigations and solve problems.

Standard 13: Mathematics as Communication

In the program, prospective teachers learn to communicate their thinking clearly and coherently to others using appropriate language, symbols and technologies. Prospective teachers develop communication skills in conjunction with mathematical literacy in each major component of a subject matter program.

Required Elements:

13.1 Articulate mathematical ideas verbally and in writing, using appropriate terminology.

13.2 Where appropriate present mathematical explanations suitable to a variety of grade levels.

13.3 Present mathematical information in various forms, including but not limited to models, charts, graphs, tables, figures, and equations.

13.4 Analyze and evaluate the mathematical thinking and strategies of others.

13.5 Use clarifying and extending questions to learn and to communicate mathematical ideas.

13.6 Use appropriate technologies to present mathematical ideas and concepts.

Standard 14: Reasoning

In the program, prospective teachers of mathematics learn to understand that reasoning is fundamental to knowing and doing mathematics. Reasoning and proof accompany all mathematical activities in the program.

Required Elements:

14.1 Formulate and test conjunctures using inductive reasoning, construct counter-examples, make valid deductive arguments, and judge the validity of mathematical arguments in each content domain of the subject matter requirements.

14.2 Present informal and formal proofs in oral and written formats in each content domain of the subject matter requirements.

Standard 15: Mathematical Connections

In the program, prospective teachers of mathematics develop a view of mathematics as an integrated whole, seeing connections across different mathematical content areas. Relationships among mathematical subjects and applications are a consistent theme of the subject matter program's curriculum.

Required Elements:

15.1 Illustrate, when possible, abstract mathematical concepts using applications.

15.2 Investigate ways mathematical topics are inter-related.

15.3 Apply mathematical thinking and modeling to solve problems that arise in other disciplines.

15.4 Recognize how a given mathematical model can represent a variety of situations.

15.5 Create a variety of models to represent a single situation.

15.6 Understand the interconnectedness of topics in mathematics from an historical perspective.

Standard 16: Delivery of Instruction

In the program, faculty use multiple instructional strategies, activities and materials that are appropriate for effective mathematics instruction.

Required Elements:

16.1 Is taught in a way that fosters conceptual understanding as well as procedural knowledge.

16.2 Incorporates a variety of instructional formats including but not limited to direct instruction, collaborative groups, individual exploration, peer instruction, and whole class discussion led by students.

16.3 Provides for learning mathematics in different modalities, e.g., visual, auditory, and kinesthetic.

16.4 Develops and reinforces mathematical skills and concepts through open-ended activities.

16.5 Uses a variety of appropriate technologies.

16.6 Includes approaches that are appropriate for use at a variety of grade levels.

Attachment to
Standard 11: Required Subjects of Study

The main purpose of the Subject Matter Requirements (SMRs) is to provide a guideline for the education of prospective mathematics teachers so that they will be well equipped to teach to the state-adopted Mathematics Content Standards for California Public Schools: Kindergarten Through Grade Twelve (1997), and that they have a mathematical understanding and proficiency beyond those Standards. Taken at face value, the SMRs define a minimum core of skills, abilities, and understandings for all candidates of the Single Subject Teaching Credential in Mathematics. Ideally, teacher candidates develop an advanced viewpoint of the content areas represented in this core. The intent of this appendix is to give a sense of the mathematical context in which such advanced viewpoints can be developed. The appendix provides examples and ideas for this development, and is not intended to be prescriptive. While some of these examples may seem obvious to a professor of mathematics, many mathematics majors do not make the connections. Therefore, these ideas are important for prospective teachers.

It is important to note three principles that guided the development of the SMRs:

a) mathematical reasoning is central to mathematical understanding;
b) mathematics requires knowledge that is connected and integrated; and
c) college faculty are central to shaping the curriculum of subject matter programs.

First, the emphasis on mathematical reasoning amplifies what is already clearly enunciated in a critical passage of the Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve (1999; Framework):

> From kindergarten through grade 7, these [content] standards have impressed on the students the importance of logical reasoning in mathematics. Starting with grade 8, students should be ready for the basic message that logical reasoning is the underpinning of all of mathematics. In other words, every assertion can be justified by logical deduction from previously known facts. Students should begin to learn to prove every statement that they make. Every textbook or mathematics lesson should strive to convey this message, and to convey it well. (p. 154)

In order for such a vision of mathematics education to materialize, teachers themselves need to be well versed in writing proofs and explaining them. For this reason, the SMRs emphasize logical explanations, and formal and informal proofs. Explanations and proofs also underscore the fact that logical arguments occur not only in Euclidean geometry but everywhere.

A proof is a logical explanation of why a statement holds. It need not have any particular form, and the emphasis should be on the student understanding why a result holds. Written proofs in textbooks may serve as a model for exposition, but never as a model for the discovery of a proof. Proofs are usually found by painstaking trials and errors, and almost never in the logical sequence of steps laid out in written proofs. It should be emphasized that it is the logical correctness of a proof that is important, not the literary polish of the presentation of the proof. The common complaint that geometry proofs in a real classroom have become a ritual divorced from mathematics would disappear if teachers are made more aware of the need to pay attention to mathematical substance rather than minute details of the write-up of a proof. A correct proof can be legitimately presented in many ways (e.g., two-column format, paragraph format, flow-chart format). No one format is inherently superior to any other.

Second, the integration of subject matter is implied in more than a few of the standards. Although the SMRs are divided into separate content domains (e.g., algebra, geometry) such a division is more for the convenience of communication rather than an advocacy for a rigid separation of mathematical instruction. For example, prospective teachers should be able to analyze and solve polynomial equations using the roots of unity. This statement assumes that the prospective teacher understands De Moivre's Theorem (SMR 5.1e) and basic properties of regular polygons. In this case, algebra, trigonometry, and geometry are completely intermingled. As another example, prospective teachers need to be able to teach the graphing of polynomials, but simple facts about such graphs (e.g., that the graph of an *nth* degree polynomial has at most n-1 "peaks" and "valleys") are not accessible without the use of calculus.

Third, the SMRs are not prescriptive about curriculum or pedagogy. There is plenty of room for the creative and informed judgements of faculty to direct the education of teachers of mathematics. For example, although it is not included in the SMRs, faculty may choose to present the derivation of the cubic formula for the purpose of deepening teachers' understanding and appreciation of the quadratic formula. Similarly, some faculty may view SMR 1.3c, which deals with properties of the logarithm function, as an implicit invitation to go into the origin of the logarithm. Napier’s invention of logarithms in the 1600s was the device which, in the word of the French mathematician-astronomer Laplace, "by shortening the labors, doubled the life of the astronomer.” When teachers understand this utility, and the parallels of the discovery of logarithms with the discovery and development of computing technologies, they are much better equipped to motivate students’ understanding of such mathematical topics.

The following sections provide some ideas and examples for developing an advanced viewpoint, particularly about the importance of mathematical reasoning and connections, through the main subject areas of the SMRs.

ALGEBRA

Mathematical reasoning

Prospective teachers' understanding of the three fields they use most often – rational, real, and complex numbers – should include what it means for rational and real numbers to be ordered fields, and why complex numbers cannot be ordered. Inequalities make sense in real numbers, because they are ordered. However, prospective teachers should understand that although inequalities do not make sense in complex numbers, equations have a fuller role with them, because every polynomial equation with real or complex coefficients can be completely solved in complex numbers by the Fundamental Theorem of Algebra (SMR 1.1c, 1.2c).

Implicit in SMR 1.2a, which calls for a proof of why the graph of a linear inequality is a half plane, is the need for a proof of the fact that the graph of a linear function is a straight line. The latter proof requires the use of basic properties of similar triangles.

The proof of the result that the roots of real polynomials come in complex conjugate pairs (SMR 1.2b) allows one to see how to make use of the Fundamental Theorem of Algebra in a nontrivial way. In the process, one gains a better understanding of both the Fundamental Theorem of Algebra and the Quadratic Formula.

The rational root theorem for polynomials with integer coefficients (SMR 1.2b) is one that students and textbooks often mistake as a recipe for locating all the roots of such a polynomial. By reviewing the proof carefully, a prospective teacher is likely to understand the full meaning of this theorem.

The Binomial Theorem (SMR 1.2b) occupies a place of honor in algebra and has important connections in other areas of mathematics. Prospective teachers should be able to understand one of its most accessible proofs, and thereby learn a substantive application of mathematical induction.

Connections

Although the SMRs are organized into discrete content domains (e.g., algebra or calculus), prospective teachers should learn that these domains cannot be rigidly separated. For example, the importance of the exponential function (SMR 1.3c) stems primarily from the fact that it is the unique solution of the differential equation $f'(x) = f(x)$ with the initial condition $f(0) = 1$ (SMR 5.3f). It should be emphasized that it is because of this differential equation that the exponential function e^x (exp x) shows up in the growth and decay problems of algebra textbooks.

The fundamental difference between polynomial functions and both exponential and logarithmic functions should be emphasized (SMR 1.3b, c). The overriding concern with a polynomial is to locate its roots and the roots of its derivative (to get the x-intercepts as well as the "peaks" and "valleys" of its graph). For exponential and logarithmic functions, however, such a concern does not exist because log x has exactly one root whereas exp x has no root at all. Moreover, both are strictly increasing functions; their graphs have no "peaks" or "valleys." Therefore our interests in the latter functions are different in kind. Our interests in the exponential and logarithmic functions are that log x converts multiplication into addition [i.e., $\log (ab) = \log a + \log b$] while exp x does the opposite [i.e., $\exp (a+b) = (\exp a)(\exp b)$], and the fact that they are inverses to each other [i.e., $\log (\exp x) = x$ for all x and $\exp (\log y) = y$ for all positive y]. The algebraic properties of log x account for its historical importance as a computational aid (logarithm tables). Analytically, it is the fact that exp x is the solution of $f'(x) = f(x)$, as discussed above, and that log x is the function that has derivative $1/x$ and satisfies $\log 1 = 0$. The trigonometric functions are important for yet a different reason: periodicity (SMR 5.1c). Many natural phenomena are periodic, and their modeling would require the trigonometric functions. Such a conceptual understanding of these three classes of functions is indispensable to helping teachers make sense of the functions they see almost daily in algebra classes.

Although the topic of rationalizing denominators is not one that is seen as essential, it is one for which a strong connection can be made with ideas from an advanced perspective. One example that shows how rationalizing denominators is related to more advanced ideas is the "rationalizing" of the denominator of $\frac{1}{4^{\frac{1}{3}} - 2(2^{\frac{1}{3}}) + 2}$, which is to find a polynomial in $2^{\frac{1}{3}}$ with rational coefficients so that multiplying the denominator $4^{\frac{1}{3}} - 2(2^{\frac{1}{3}}) + 2$ by this polynomial equals a rational number. Let $x = 2^{\frac{1}{3}}$, then the denominator becomes $x^2 - 2x + 2$. In the polynomial ring, Q[x] (where Q is the field of rational numbers), the polynomials $x^3 - 2$ and $x^2 - 2x + 2$ are relatively prime and therefore, by the Euclidean algorithm, there are polynomials p(x) and q(x) in Q[x] so that p(x)($x^2 - 2x + 2$) + q(x)($x^3 - 2$) = 1. Letting $x = 2^{\frac{1}{3}}$ gives p($2^{\frac{1}{3}}$)($4^{\frac{1}{3}}$ -2($2^{\frac{1}{3}}$) + 2) = 1. It turns out that p(x) = $\frac{1}{10}(x^2 + 3x + 4)$, so that multiplying the numerator and denominator of $\frac{1}{4^{\frac{1}{3}} - 2(2^{\frac{1}{3}}) + 2}$ by p($2^{\frac{1}{3}}$) = $\frac{1}{10}$($4^{\frac{1}{3}}$ + 3($2^{\frac{1}{3}}$) + 4) leads to $\frac{1}{4^{\frac{1}{3}} - 2(2^{\frac{1}{3}}) + 2} = \frac{1}{10}$ ($4^{\frac{1}{3}}$ + 3($2^{\frac{1}{3}}$) + 4). Engaging in this example will help candidates to make a good connection between topics that they studied in their abstract algebra course and ideas related to the high school curriculum.

GEOMETRY

Mathematical Reasoning

The great challenge in a college geometry course for prospective teachers is teaching fluency with informal and formal proofs of geometric theorems in general and theorems in Euclidean geometry (SMR 2.2) in particular. There is a thorough discussion of this issue in Chapter 3 of the 1999 Framework (pp. 162-7; see also Appendix D on pp. 279-296). The following are key points:

(a) One cannot learn how to prove theorems in geometry without any geometric intuition. One way to acquire such an intuition is to perform constructions with a ruler and compass, and to examine many models of standard solids (e.g., cubes, cones, cylinders).

(b) An introductory college geometry course should start from the beginning. One way to gain the confidence of prospective teachers is not to force them to write any proofs until they have been shown many nontrivial proofs of interesting theorems (see Appendix D of the 1999 Framework). Begin slowly, allowing them to imitate some standard proofs before they venture forth on their own. This is analogous to the method of teaching people how to speak a foreign language whereby you have them listen to the language for many hours before asking them to try to speak it.

(c) In middle and high school geometry as well as college-level geometry courses, one should de-emphasize the proofs of simple theorems that come near the beginning of the axiomatic development. The proofs of such theorems are harder to learn than those of theorems that follow, and this is true not only for beginners but also for professional mathematicians as well. These proofs also tend to be tedious and uninspiring. One way to acquaint prospective teachers with the proofs of more substantive theorems as soon as possible is to adopt the method of "local axiomatics," which is to list the facts one needs for a particular proof, and then proceed to construct the proof on the basis of these facts. This approach mirrors the axiomatic method because, in effect, these facts are the "axioms" in this particular setting (see the examples in Appendix D of the 1999 Framework).

Connections

The historical importance of the parallel postulate, not just in geometry but in all of mathematics up to the nineteenth century, should be thoroughly discussed (SMR 2.1a, b). In middle and high school geometry textbooks, this postulate is stated (if it is stated at all) as "through a point not on a given line, there is one and only one line parallel to the given line." The correct formulation replaces the phrase "there is one and only one" with "there is at most one." In other words, while the existence of the parallel line can be proved, the uniqueness must be assumed. This then gives a natural setting to introduce the concept of "uniqueness," which is a difficult concept for many students. In this context, an informal discussion of the counterparts of the parallel postulate in spherical and hyperbolic geometry (SMR 2.1b) will likely clarify the situation.

The deduction of the parallel postulate from the assumption that "every triangle has an angle sum of $180°$" is somewhat more sophisticated than most of the theorems in plane Euclidean geometry, but when done carefully it can be immensely rewarding (SMR 2.1a).

Although the notion of area will be defined using the Riemann integral in the context of calculus (SMR 5.4d), it is essential for the teaching of middle and high school geometry that a basic definition of area be provided for plane geometric figures. From this definition, a prospective teacher should be able to derive the area formulas for regular polygons, and many other plane geometric figures.

The theorem that every polygon can be triangulated into nonoverlapping triangles allows the areas of polygons to be calculated once the areas of the triangles are known (SMR 2.2c). There is, however, no analogous theorem for the volume of a general polyhedron (SMR 2.3b). This is because it can be proved (using advanced techniques) that there is no corresponding elementary algorithm to compute the volume of a general (non-regular) tetrahedron from the volume of a cube. Although the proof of this theorem is too difficult for an introductory course, prospective teachers need to know this fact to be able to explain to their students why all volume formulas (except that of a rectangular prism) require the use of calculus or equivalent limit arguments. However, from a basic definition of volume, with the use of informal arguments and Cavalieri's Principle, the volumes of prisms, pyramids, cones, cylinders, and spheres can be informally derived. Moreover, teachers should be aware that formally, the coefficient 1/3 in the volume formulas of cones and pyramids comes from integrating x^2 (SMR 5.4d).

A key reason for introducing coordinates and discussing geometric transformations (SMR 2.4a, b) is to be able to clarify the concepts of congruence and similarity, not just for triangles or polygons, but for all plane and space figures. In other words, one defines two such figures to be *congruent* if one is the image of the other under an isometry, and defines them to be *similar* if one is the image of the other under an isometry followed by a dilation. Then it can be shown that when the figures are polygons, these concepts coincide with those of the equality of angles and proportionality of sides.

NUMBER THEORY

Mathematical Reasoning

The well known divisibility rules for division by 3, 4, 5, 8, or 9 are usually stated and used in middle and high school textbooks but not often explained. It is imperative that prospective teachers understand the simple proofs of these rules (SMR 3.1a).

From the point of view of middle and high school mathematics, there are at least two aspects of the Fundamental Theorem of Arithmetic that are noteworthy. First, a completely correct proof of the existence of a prime decomposition for whole numbers requires the use of complete induction (and this gives an important example of a different application of mathematical induction). Second, whereas in middle and high school mathematics only the existence part of the theorem is used, one discovers that in fact it is the uniqueness of the prime decomposition that is important and difficult to prove. Experience shows that this particular uniqueness statement - more so than the uniqueness in the parallel postulate or the uniqueness of the remainder in the division algorithm - is elusive to beginners. The uniqueness is an essential aspect of the Fundamental Theorem of Arithmetic; otherwise, the proof of the irrationality of 5 (or any whole number not a perfect square) or why every fraction is equivalent to a unique fraction in lowest terms would be meaningless.

Connections

The Euclidean algorithm (SMR 3.1c) requires a strong understanding of the division algorithm, including a clear conceptualization of a remainder, and thus the uniqueness of the remainder in the division algorithm. This is another area in which the content domains merge. Prospective teachers should understand both the division algorithm and the Euclidean algorithm for polynomials with real coefficients, and the relationship to the results in number theory.

CALCULUS

Mathematical Reasoning

One should emphasize that the sine and cosine addition theorems are the defining theorems of trigonometry (SMR 5.1b). Indeed, it can be proved that sine and cosine are the only differentiated functions satisfying the addition theorems and the condition that sin 0 = 0 and cos 0 = 1. Moreover, every trigonometric identity is a consequence of these addition theorems, and the identity that $\sin^2 x + \cos^2 x = 1$. Thus the latter identity and the addition theorems are the foundation of trigonometry. This fact gives structure to the subject, and should be clearly understood by each prospective teacher.

In the teaching of calculus, it would be inappropriate to insist on epsilon-delta proofs, but it would be equally inappropriate to eliminate such proofs altogether. Therefore, SMR 5.2 requires that at least the correct definition of limit be provided and applied in a restricted way. This can be accomplished by proving the continuity of quadratic polynomials using epsilon-delta. One benefit of this insistence on a minimal amount of rigor is to expose prospective teachers to the fallacy of the common perception that the continuity of f(x) means "a small change in x produces a small change in f(x)." For instance, if this were the case, should not a change in x to the order of 1/10000 produce a "small" change in f(x)? The answer is, of course, no, because if f(x) = $10^9 x$, then a change in x of 1/10000 produces a change of 100000 in f(x). Thus, one can see why precision in mathematics (such as that found in the tortuous definition of continuity) is necessary. Not insisting on precise proofs on the most common differentiation formulas is likely to invite some abuse. For example, the usual proof "from the product rule of differentiation, one can prove the quotient rule" is a common pitfall that should be avoided, especially in the context of middle and high school mathematics. The putative proof goes as follows: because f(x) (1/f(x)) = 1, differentiating both sides and applying the product rule on the right side of the formula gives f'(x) (1/f(x)) + f(x) (1/f(x))' = 0, from which it follows that (1/f(x))' = - f'(x)/[f(x)^2]. Once this is known, another application of the product rule to g(x)(1/f(x)) gives the usual quotient rule for g(x)/f(x). This is the "proof" of the quotient rule. The fallacy of the preceding argument lies in the fact that until one knows 1/f(x) is differentiated one cannot apply the product rule to f(x)(1/f(x)). Of course, when one tries to prove the differentiation of 1/f(x), the result is the usual messy proof of the quotient rule. What can be claimed is that the above method gives a mnemonic device to remember the quotient rule. Such a statement, when so carefully phrased, has pedagogical value in a calculus classroom, but by no means should one convey the misconception that the product rule proves the quotient rule. Similar comments apply to the differentiation of the square root of a function or, in fact, of any rational power of a function.

The calculus SMRs require the proofs of few theorems, one of which is the proof of the Fundamental Theorem of Calculus (SMR 5.4c). Intended by this SMR is a proof that assumes the basic properties of continuous functions and the integral (e.g., that a continuous function attains a maximum and a minimum on a closed interval, that the integral is linear in the integrand, and that the integral of positive functions is positive). The reason prospective teachers should know this proof is not only that the Fundamental Theorem is truly fundamental (and why this is so should, of course, be carefully explained), but also that this proof is very instructive.

Connections

Both finite and infinite geometric series are important because they appear frequently (SMR 5.5a). In particular, one aspect of infinite geometric series deserves comment, namely the fact that the formal way of summing a geometric series gives rise to the expression of a repeating decimal as a fraction. This mechanism should be conducted carefully as it is often presented incorrectly in middle and high school textbooks. One reason for mentioning the convergence of infinite geometric series (SMR 5.5b) is to make sense of infinite decimals: an infinite decimal is merely a shorthand notation for a particular kind of infinite series. For Taylor series (SMR 5.5c), candidates should know at least the formalism of associating a power series to any one of the elementary functions. Candidates should be able to recognize the sine, cosine, and exponential series.

HISTORY OF MATHEMATICS

Many important developments in mathematics are too advanced to be discussed in an introductory course on the history of mathematics, yet four major developments that directly impact middle and high school mathematics deserve special attention (SMR 6.1b). The first development is the history of numeral systems through the early civilizations of Babylon, Rome, and China, and through the so-called Hindu-Arabic decimal system. A second development is the evolution of symbolic algebra, which includes contributions from Diophantus, the Hindus, Viete, and the finishing touches of Descartes. An understanding of this long and uneasy development enhances one's understanding of middle and high school mathematics as a whole. The third development is of calculus, which is rooted in ideas from Eudoxus and Archimedes, the rich but informal development of Newton and Leibniz, and the rigorous formulation that culminated with Cauchy. The fourth and last development is the concept of a proof and, therewith, the concept of an axiomatic system. Proofs formally originated with Euclid's work, and until the twentieth century, were essentially the defining characteristic of European mathematics. For almost two centuries, the questionable foundation of calculus almost forced an abandonment of the classical ideal of proofs in mathematics. It was only toward the end of the nineteenth century when proofs would again occupy center stage and a clear definition of a proof was achieved.

Mathematics Subject Matter Requirements [2]

Part I: Content Domains for Subject Matter Understanding and Skill in Mathematics

Domain 1. Algebra

Candidates demonstrate an understanding of the foundations of the algebra contained in the Mathematics Content Standards for California Public Schools: Kindergarten Through Grade Twelve (1997) as outlined in the Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve (1999) from an advanced standpoint. To ensure a rigorous view of algebra and its underlying structures, candidates have a deep conceptual knowledge. They are skilled at symbolic reasoning and use algebraic skills and concepts to model a variety of problem-solving situations. They understand the power of mathematical abstraction and symbolism.

1.1 Algebraic Structures

a. Know why the real and complex numbers are each a field, and that particular rings are not fields (e.g., integers, polynomial rings, matrix rings)
b. Apply basic properties of real and complex numbers in constructing mathematical arguments (e.g., if $a < b$ and $c < 0$, then $ac > bc$)
c. Know that the rational numbers and real numbers can be ordered and that the complex numbers cannot be ordered, but that any polynomial equation with real coefficients can be solved in the complex field

(Mathematics Content Standards for California Public Schools, Grade 6, Number Sense: 1.0, 2.0; Grade 7, Algebra and Functions: 1.0; Algebra I: 1.0, 3.0-7.0, 9.0-15.0, 24.0, 25.0; Geometry: 1.0, 17.0; Algebra II: 1.0-8.0, 11.0, 24.0, 25.0; Trigonometry: 17.0; Mathematical Analysis: 2.0; Linear Algebra: 9.0, 11.0)

1.2 Polynomial equations and Inequalities

a. Know why graphs of linear inequalities are half planes and be able to apply this fact (e.g., linear programming)
b. Prove and use the following:
   - The Rational Root Theorem for polynomials with integer coefficients
   - The Factor Theorem
   - The Conjugate Roots Theorem for polynomial equations with real coefficients
   - The Quadratic Formula for real and complex quadratic polynomials
   - The Binomial Theorem
c. Analyze and solve polynomial equations with real coefficients using the Fundamental Theorem of Algebra

[2] The Subject Matter Requirements are complemented by the **Attachment to Standard 11**, starting on page 19

(Mathematics Content Standards for California Public Schools, Grade 7, Algebra and Functions: 2.0-4.0; Algebra I: 1.0, 2.0, 4.0-10.0, 12.0-15.0, 17.0-23.0; Algebra II: 2.0-11.0, 16.0, 17.0; Trigonometry: 17.0, 18.0; Mathematical Analysis: 4.0, 6.0)

1.3 Functions

a. Analyze and prove general properties of functions (i.e., domain and range, one-to-one, onto, inverses, composition, and differences between relations and functions)
b. Analyze properties of polynomial, rational, radical, and absolute value functions in a variety of ways (e.g., graphing, solving problems)
c. Analyze properties of exponential and logarithmic functions in a variety of ways (e.g., graphing, solving problems)

(Mathematics Content Standards for California Public Schools, Grade 6, Algebra and Functions: 1.0; Grade 7, Number Sense: 1.0, 2.0; Algebra and Functions: 3.0; Algebra I: 3.0-6.0, 10.0, 13.0, 15.0-18.0, 21.0-23.0; Algebra II: 1.0-4.0, 6.0-17.0, 24.0, 25.0; Trigonometry: 2.0, 4.0-8.0, 19.0; Mathematical Analysis: 6.0, 7.0; Calculus: 9.0)

1.4 Linear Algebra

a. Understand and apply the geometric interpretation and basic operations of vectors in two and three dimensions, including their scalar multiples and scalar (dot) and cross products
b. Prove the basic properties of vectors (e.g., perpendicular vectors have zero dot product)
c. Understand and apply the basic properties and operations of matrices and determinants (e.g., to determine the solvability of linear systems of equations)

(Mathematics Content Standards for California Public Schools, Algebra I: 9.0; Algebra II: 2.0; Mathematical Analysis: 1.0; Linear Algebra: 1.0-12.0)

Domain 2. Geometry

Candidates demonstrate an understanding of the foundations of the geometry contained in the Mathematics Content Standards for California Public Schools: Kindergarten Through Grade Twelve (1997) as outlined in the Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve (1999) from an advanced standpoint. To ensure a rigorous view of geometry and its underlying structures, candidates have a deep conceptual knowledge. They demonstrate an understanding of axiomatic systems and different forms of logical arguments. Candidates understand, apply, and prove theorems relating to a variety of topics in two- and three-dimensional geometry, including coordinate, synthetic, non-Euclidean, and transformational geometry.

2.1 Parallelism

a. Know the Parallel Postulate and its implications, and justify its equivalents (e.g., the Alternate Interior Angle Theorem, the angle sum of every triangle is 180 degrees)
b. Know that variants of the Parallel Postulate produce non-Euclidean geometries (e.g., spherical, hyperbolic)

(Mathematics Content Standards for California Public Schools, Algebra I: 8.0, 24.0; Geometry: 1.0-3.0, 7.0, 13.0)

2.2 Plane Euclidean Geometry

a. Prove theorems and solve problems involving similarity and congruence
b. Understand, apply, and justify properties of triangles (e.g., the Exterior Angle Theorem, concurrence theorems, trigonometric ratios, Triangle Inequality, Law of Sines, Law of Cosines, the Pythagorean Theorem and its converse)
c. Understand, apply, and justify properties of polygons and circles from an advanced standpoint (e.g., derive the area formulas for regular polygons and circles from the area of a triangle)
d. Justify and perform the classical constructions (e.g., angle bisector, perpendicular bisector, replicating shapes, regular n-gons for n equal to 3, 4, 5, 6, and 8)
e. Use techniques in coordinate geometry to prove geometric theorems

(Mathematics Content Standards for California Public Schools, Grade 6, Algebra and Functions: 2.0, 3.0; Measurement and Geometry: 2.0; Grade 7, Measurement and Geometry: 1.0-3.0; Algebra I: 8.0, 24.0; Geometry: 1.0-6.0, 8.0-16.0, 18.0-21.0; Algebra II: 16.0, 17.0; Trigonometry: 12.0-14.0, 18.0, 19.0; Mathematical Analysis: 5.0)

2.3 Three-Dimensional Geometry

a. Demonstrate an understanding of parallelism and perpendicularity of lines and planes in three dimensions
b. Understand, apply, and justify properties of three-dimensional objects from an advanced standpoint (e.g., derive the volume and surface area formulas for prisms, pyramids, cones, cylinders, and spheres)

(Mathematics Content Standards for California Public Schools, Grade 6, Measurement and Geometry: 1.0; Grade 7, Measurement and Geometry: 2.0; Algebra I: 24.0; Geometry: 2.0, 3.0, 12.0, 17.0; Mathematical Analysis: 5.0)

2.4 Transformational Geometry

a. Demonstrate an understanding of the basic properties of isometries in two- and three-dimensional space (e.g., rotation, translation, reflection)
b. Understand and prove the basic properties of dilations (e.g., similarity transformations or change of scale)

(Mathematics Content Standards for California Public Schools, Geometry: 11.0, 22.0)

Domain 3. Number Theory

Candidates demonstrate an understanding of the number theory and a command of the number sense contained in the Mathematics Content Standards for California Public Schools: Kindergarten Through Grade Twelve (1997) as outlined in the Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve (1999) from an advanced standpoint. To ensure a rigorous view of number theory and its underlying structures, candidates have a deep conceptual knowledge. They prove and use properties of natural numbers. They formulate conjectures about the natural numbers using inductive reasoning, and verify conjectures with proofs.

3.1 Natural Numbers

a. Prove and use basic properties of natural numbers (e.g., properties of divisibility)
b. Use the Principle of Mathematical Induction to prove results in number theory
c. Know and apply the Euclidean Algorithm
d. Apply the Fundamental Theorem of Arithmetic (e.g., find the greatest common factor and the least common multiple, show that every fraction is equivalent to a unique fraction where the numerator and denominator are relatively prime, prove that the square root of any number, not a perfect square number, is irrational)

(Mathematics Content Standards for California Public Schools, Grade 6, Number Sense: 2.0; Grade 7, Number Sense: 1.0; Algebra I: 1.0, 2.0, 12.0, 24.0, 25.0; Geometry: 1.0; Algebra II: 21.0, 23.0, 25.0; Mathematical Analysis: 3.0)

Domain 4. Probability and Statistics

Candidates demonstrate an understanding of the statistics and probability distributions for advanced placement statistics contained in the Mathematics Content Standards for California Public Schools: Kindergarten Through Grade Twelve (1997) as outlined in the Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve (1999) from an advanced standpoint. To ensure a rigorous view of probability and statistics and their underlying structures, candidates have a deep conceptual knowledge. They solve problems and make inferences using statistics and probability distributions.

4.1 Probability

a. Prove and apply basic principles of permutations and combinations
b. Illustrate finite probability using a variety of examples and models (e.g., the fundamental counting principles)
c. Use and explain the concept of conditional probability
d. Interpret the probability of an outcome
e. Use normal, binomial, and exponential distributions to solve and interpret probability problems

(Mathematics Content Standards for California Public Schools, Grade 6, Statistics, Data Analysis, and Probability: 3.0; Algebra II: 18.0-20.0; Probability and Statistics: 1.0-4.0; Advanced Probability and Statistics: 1.0-4.0, 7.0, 9.0, 17.0, 18.0)

4.2 Statistics

a. Compute and interpret the mean, median, and mode of both discrete and continuous distributions
b. Compute and interpret quartiles, range, variance, and standard deviation of both discrete and continuous distributions

c. Select and evaluate sampling methods appropriate to a task (e.g., random, systematic, cluster, convenience sampling) and display the results
d. Know the method of least squares and apply it to linear regression and correlation
e. Know and apply the chi-square test

(Mathematics Content Standards for California Public Schools, Grade 6, Statistics, Data Analysis, and Probability: 1.0, 2.0; Grade 7, Statistics, Data Analysis, and Probability: 1.0; Probability and Statistics: 5.0-7.0; Advanced Probability and Statistics: 4.0-6.0, 8.0, 10.0-13.0, 15.0-17.0, 19.0)

Domain 5. Calculus

Candidates demonstrate an understanding of the trigonometry and calculus contained in the Mathematics Content Standards for California Public Schools: Kindergarten Through Grade Twelve (1997) as outlined in the Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve (1999) from an advanced standpoint. To ensure a rigorous view of trigonometry and calculus and their underlying structures, candidates have a deep conceptual knowledge. They apply the concepts of trigonometry and calculus to solving problems in real-world situations.

5.1 Trigonometry

a. Prove that the Pythagorean Theorem is equivalent to the trigonometric identity $\sin^2 x + \cos^2 x = 1$ and that this identity leads to $1 + \tan^2 x = \sec^2 x$ and $1 + \cot^2 x = \csc^2 x$
b. Prove the sine, cosine, and tangent sum formulas for all real values, and derive special applications of the sum formulas (e.g., double angle, half angle)
c. Analyze properties of trigonometric functions in a variety of ways (e.g., graphing and solving problems)
d. Know and apply the definitions and properties of inverse trigonometric functions (i.e., arcsin, arccos, and arctan)
e. Understand and apply polar representations of complex numbers (e.g., DeMoivre's Theorem)

(Mathematics Content Standards for California Public Schools, Algebra I: 24.0; Geometry: 3.0, 14.0, 18.0, 19.0; Algebra II: 24.0, 25.0; Trigonometry: 1.0-6.0, 8.0-11.0, 19.0; Mathematical Analysis: 1.0, 2.0; Calculus: 18.0, 20.0)

5.2 Limits and Continuity

a. Derive basic properties of limits and continuity, including the Sum, Difference, Product, Constant Multiple, and Quotient Rules, using the formal definition of a limit
b. Show that a polynomial function is continuous at a point
c. Know and apply the Intermediate Value Theorem, using the geometric implications of continuity

(Mathematics Content Standards for California Public Schools, Algebra I: 24.0; Geometry: 3.0; Algebra II: 1.0, 15.0; Mathematical Analysis: 8.0; Calculus: 1.0-4.0)

5.3 Derivatives and Applications

a. Derive the rules of differentiation for polynomial, trigonometric, and logarithmic functions using the formal definition of derivative
b. Interpret the concept of derivative geometrically, numerically, and analytically (i.e., slope of the tangent, limit of difference quotients, extrema, Newton's method, and instantaneous rate of change)
c. Interpret both continuous and differentiable functions geometrically and analytically and apply Rolle's Theorem, the Mean Value Theorem, and L'Hopital's rule
d. Use the derivative to solve rectilinear motion, related rate, and optimization problems
e. Use the derivative to analyze functions and planar curves (e.g., maxima, minima, inflection points, concavity)
f. Solve separable first-order differential equations and apply them to growth and decay problems

(<u>Mathematics Content Standards for California Public Schools</u>, Algebra I: 5.0-8.0, 10.0, 11.0, 13.0, 21.0, 23.0; Geometry: 3.0; Algebra II: 1.0, 9.0, 10.0, 12.0, 15.0; Trigonometry: 7.0, 15.0-19.0; Mathematical Analysis: 5.0, 7.0; Calculus: 1.0, 4.0-12.0, 27.0)

5.4 Integrals and Applications

a. Derive definite integrals of standard algebraic functions using the formal definition of integral
b. Interpret the concept of a definite integral geometrically, numerically, and analytically (e.g., limit of Riemann sums)
c. Prove the Fundamental Theorem of Calculus, and use it to interpret definite integrals as antiderivatives
d. Apply the concept of integrals to compute the length of curves and the areas and volumes of geometric figures

(<u>Mathematics Content Standards for California Public Schools</u>, Algebra I: 24.0; Geometry: 9.0; Calculus: 13.0-23.0)

5.5 Sequences and Series

a. Derive and apply the formulas for the sums of finite arithmetic series and finite and infinite geometric series (e.g., express repeating decimals as a rational number)
b. Determine convergence of a given sequence or series using standard techniques (e.g., Ratio, Comparison, Integral Tests)
c. Calculate Taylor series and Taylor polynomials of basic functions

(<u>Mathematics Content Standards for California Public Schools</u>, Algebra I: 24.0, 25.0; Algebra II: 21.0-23.0; Mathematical Analysis: 8.0; Calculus: 23.0-26.0)

Domain 6. History of Mathematics

Candidates understand the chronological and topical development of mathematics and the contributions of historical figures of various times and cultures. Candidates know important mathematical discoveries and their impact on human society and thought. These discoveries form a historical context for the content contained in the Mathematics Content Standards for California Public Schools: Kindergarten Through Grade Twelve (1997) as outlined in the Mathematics Framework for California Public Schools: Kindergarten Through Grade Twelve (1999; e.g., numeration systems, algebra, geometry, calculus).

6.1 Chronological and Topical Development of Mathematics

a. Demonstrate understanding of the development of mathematics, its cultural connections, and its contributions to society

b. Demonstrate understanding of the historical development of mathematics, including the contributions of diverse populations as determined by race, ethnicity, culture, geography, and gender

Part II: Subject Matter Skills and Abilities Applicable to the Content Domains in Mathematics

Candidates for Single Subject Teaching Credentials in mathematics use inductive and deductive reasoning to develop, analyze, draw conclusions, and validate conjectures and arguments. As they reason, they use counterexamples, construct proofs using contradictions, and create multiple representations of the same concept. They know the interconnections among mathematical ideas, and use techniques and concepts from different domains and sub-domains to model the same problem. They explain mathematical interconnections with other disciplines. They are able to communicate their mathematical thinking clearly and coherently to others, orally, graphically, and in writing, through the use of precise language and symbols.

Candidates solve routine and complex problems by drawing from a variety of strategies while demonstrating an attitude of persistence and reflection in their approaches. They analyze problems through pattern recognition and the use of analogies. They formulate and prove conjectures, and test conclusions for reasonableness and accuracy. They use counterexamples to disprove conjectures.

Candidates select and use different representational systems (e.g., coordinates, graphs). They understand the usefulness of transformations and symmetry to help analyze and simplify problems. They make mathematical models to analyze mathematical structures in real contexts. They use spatial reasoning to model and solve problems that cross disciplines.

(Mathematics Content Standards for California Public Schools, Grade 6, Mathematical Reasoning: 1.0-3.0; Grade 7, Mathematical Reasoning: 1.0-3.0)

Appendix A
Assembly Bill No. 537

CHAPTER 587

An act to amend Sections 200, 220, 66251, and 66270 of, to add Section 241 to, and to amend and renumber Sections 221 and 66271 of, the Education Code, relating to discrimination.

[Approved by Governor October 2, 1999. Filed with Secretary of State October 10, 1999.]

LEGISLATIVE COUNSEL'S DIGEST

AB 537, Kuehl. Discrimination.

(1) Existing law provides that it is the policy of the State of California to afford all persons in public schools and postsecondary institutions, regardless of their sex, ethnic group identification, race, national origin, religion, or mental or physical disability, equal rights and opportunities in the educational institutions of the state.

Existing law makes it a crime for a person, whether or not acting under color of law, to willfully injure, intimidate, interfere with, oppress, or threaten any other person, by force or threat of force, in the free exercise or enjoyment of any right or privilege secured to him or her by the Constitution or laws of this state or by the Constitution or laws of the United States because of the other person's race, color, religion, ancestry, national origin, disability, gender, or sexual orientation, or because he or she perceives that the other person has one or more of those characteristics.

This bill would also provide that it is the policy of the state to afford all persons in public school and postsecondary institutions equal rights and opportunities in the educational institutions of the state, regardless of any basis referred to in the aforementioned paragraph.

(2) Existing law prohibits a person from being subjected to discrimination on the basis of sex, ethnic group identification, race, national origin, religion, color, or mental or physical disability in any program or activity conducted by any educational institution or postsecondary educational institution that receives, or benefits from, state financial assistance or enrolls students who receive state student financial aid.

This bill would also prohibit a person from being subjected to discrimination on the basis of any basis referred to in paragraph (1) in any program or activity conducted by any educational institution or postsecondary educational institution that receives, or benefits from, state financial assistance or enrolls students who receive state student financial aid.

(3) This bill would state that it does not require the inclusion of any curriculum, textbook, presentation, or other material in any program or activity conducted by an educational institution or a postsecondary educational institution and would prohibit this bill from being deemed to be violated by the omission of any curriculum, textbook, presentation, or other material in any program or activity conducted by an educational institution or a postsecondary educational institution.

To the extent that this bill would impose new duties on school districts and community college districts, it would impose a state-mandated local program.

(4) The California Constitution requires the state to reimburse local agencies and school districts for certain costs mandated by the state. Statutory provisions establish procedures for making that reimbursement, including the creation of a State Mandates Claims Fund to pay the costs of mandates that do not exceed $1,000,000 statewide and other procedures for claims whose statewide costs exceed $1,000,000.

This bill would provide that, if the Commission on State Mandates determines that the bill contains costs mandated by the state, reimbursement for those costs shall be made pursuant to these statutory provisions.

The people of the State of California do enact as follows:

SECTION 1. This bill shall be known, and may be cited, as the California Student Safety and Violence Prevention Act of 2000.

SEC. 2. (a) The Legislature finds and declares all of the following:

(1) Under the California Constitution, all students of public schools have the inalienable right to attend campuses that are safe, secure, and peaceful. Violence is the number one cause of death for young people in California and has become a public health problem of epidemic proportion. One of the Legislature's highest priorities must be to prevent our children from the plague of violence.

(2) The fastest growing, violent crime in California is hate crime, and it is incumbent upon us to ensure that all students attending public school in California are protected from potentially violent discrimination. Educators see how violence affects youth every day; they know first hand that youth cannot learn if they are concerned about their safety. This legislation is designed to protect the institution of learning as well as our students.

(3) Not only do we need to address the issue of school violence but also we must strive to reverse the increase in teen suicide. The number of teens who attempt suicide, as well as the number who actually kill themselves, has risen substantially in recent years. Teen suicides in the United States have doubled in number since 1960 and every year over a quarter of a million adolescents in the United States attempt suicide. Sadly, approximately 4,000 of these attempts every year are completed. Suicide is the third leading cause of death for youths 15 through 24 years of age. To combat this problem we must seriously examine these grim statistics and take immediate action to ensure all students are offered equal protection from discrimination under California law.

SEC. 3. Section 200 of the Education Code is amended to read:

200. It is the policy of the State of California to afford all persons in public schools, regardless of their sex, ethnic group identification, race, national origin, religion, mental or physical disability, or regardless of any basis that is contained in the prohibition of hate crimes set forth in subdivision (a) of Section 422.6 of the Penal Code, equal rights and opportunities in the educational institutions of the state. The purpose of this chapter is to prohibit acts which are contrary to that policy and to provide remedies therefor.

SEC. 4. Section 220 of the Education Code is amended to read:

220. No person shall be subjected to discrimination on the basis of sex, ethnic group identification, race, national origin, religion, color, mental or physical disability, or any basis that is contained in the prohibition of hate crimes set forth in subdivision (a) of Section 422.6 of the Penal Code in any program or activity conducted by an educational institution that receives, or benefits from, state financial assistance or enrolls pupils who receive state student financial aid.

SEC. 5. Section 221 of the Education Code is renumbered to read:
220.5. This article shall not apply to an educational institution which is controlled by a religious organization if the application would not be consistent with the religious tenets of that organization.
SEC. 6. Section 241 is added to the Education Code, to read:
241. Nothing in the California Student Safety and Violence Prevention Act of 2000 requires the inclusion of any curriculum, textbook, presentation, or other material in any program or activity conducted by an educational institution or postsecondary educational institution; the California Student Safety and Violence Prevention Act of 2000 shall not be deemed to be violated by the omission of any curriculum, textbook, presentation, or other material in any program or activity conducted by an educational institution or postsecondary educational institution.
SEC. 7. Section 66251 of the Education Code is amended to read:
66251. It is the policy of the State of California to afford all persons, regardless of their sex, ethnic group identification, race, national origin, religion, mental or physical disability, or regardless of any basis that is contained in the prohibition of hate crimes set forth in subdivision (a) of Section 422.6 of the Penal Code, equal rights and opportunities in the postsecondary institutions of the state. The purpose of this chapter is to prohibit acts that are contrary to that policy and to provide remedies therefor.
SEC. 8. Section 66270 of the Education Code is amended to read:
66270. No person shall be subjected to discrimination on the basis of sex, ethnic group identification, race, national origin, religion, color, or mental or physical disability, or any basis that is contained in the prohibition of hate crimes set forth in subdivision (a) of Section 422.6 of the Penal Code in any program or activity conducted by any postsecondary educational institution that receives, or benefits from, state financial assistance or enrolls students who receive state student financial aid.
SEC. 9. Section 66271 of the Education Code is renumbered to read:
66270.5. This chapter shall not apply to an educational institution that is controlled by a religious organization if the application would not be consistent with the religious tenets of that organization.
SEC. 10. Notwithstanding Section 17610 of the Government Code, if the Commission on State Mandates determines that this act contains costs mandated by the state, reimbursement to local agencies and school districts for those costs shall be made pursuant to Part 7 (commencing with Section 17500) of Division 4 of Title 2 of the Government Code. If the statewide cost of the claim for reimbursement does not exceed one million dollars ($1,000,000), reimbursement shall be made from the State Mandates Claims Fund.